

AP Calculus AB - Average Value, Area and Volume - Lesson

Common Core AP CALC AB 8.1, AP CALC AB 8.2, AP CALC AB 8.3, AP CALC AB 8.4, AP CALC AB 8.5, AP CALC AB 8.6, AP CALC AB 8.7, AP CALC AB 8.8, AP CALC AB 8.9, AP CALC AB 8.10, AP CALC AB 8.11, AP CALC AB 8.12 · maths-pdf.fr/en/

The average value of f on $[a, b]$ is $(1/(b - a))$ times the integral of f from a to b . Integrals also give displacement and total distance, the area between curves (top minus bottom), and volumes. Volume uses cross sections of area $A(x)$ integrated over the interval, or discs and washers when a region is revolved about an axis.

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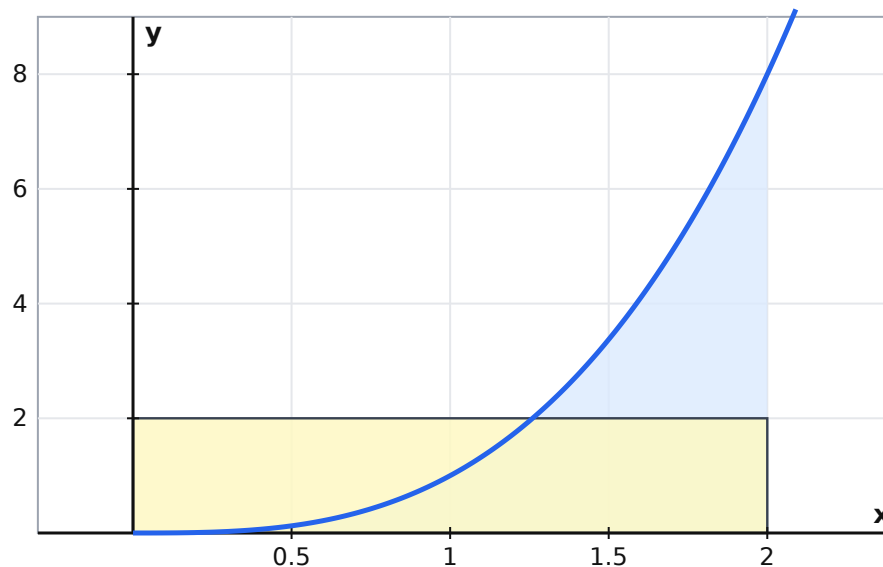
Key facts

- Average value = $(1/(b - a))$ times the integral of f over $[a, b]$.
- Displacement is the integral of v ; total distance is the integral of $|v|$.
- Area between curves = integral of (upper - lower); split the interval at every crossing.
- Volume by cross sections = integral of $A(x)$; disc method = π times the integral of (radius squared); washers subtract the inner radius squared.

How is the average value of a function defined?

Average value

$\bar{f} = \frac{\int_a^b f(x) dx}{b-a}$. Picture a rectangle on $[a, b]$ that has exactly the same area as the region under the graph; its height is the average.



Average value of x^3 on $[0, 2]$: the rectangle of height 2 has the same area as the region under the curve.

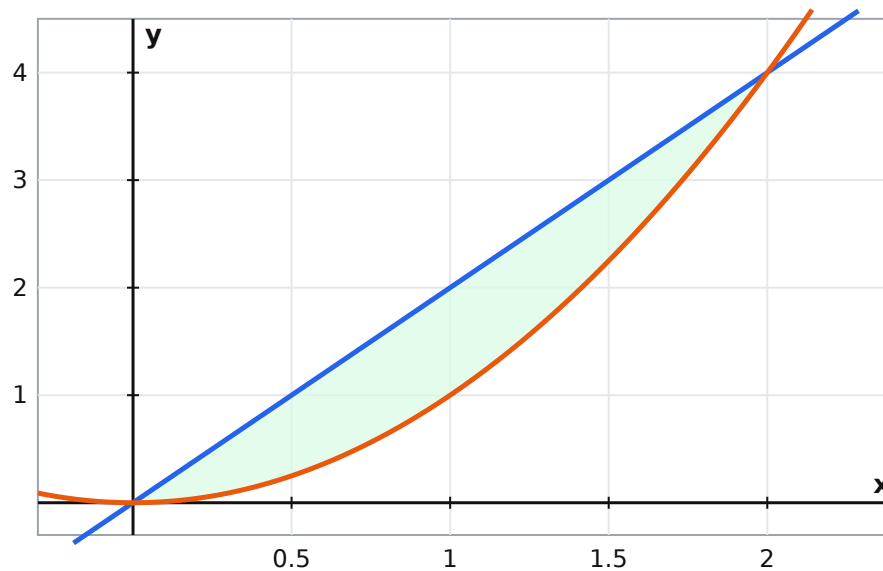
Worked case. Take $f(x) = x^3$ on $[0, 2]$: $\frac{1}{2} \int_0^2 x^3 dx = \frac{1}{2} \cdot 4 = 2$. A continuous function must reach its average value at some point in the interval (the Mean Value Theorem for Integrals); here $c^3 = 2$ puts that point at $c = \sqrt[3]{2}$.

What do integrals say about moving objects?

For a velocity function $v(t)$, the **displacement** (net change in position) equals $\int_a^b v(t) dt$, whereas the **total distance** traveled equals $\int_a^b |v(t)| dt$. Starting from a known position, $s(b) = s(a) + \int_a^b v(t) dt$. The same idea applies to any rate: integrating a rate gives the net change of the quantity.

How is the area between two curves computed?

If $f(x) \geq g(x)$ on $[a, b]$, the region has area $A = \int_a^b (f(x) - g(x)) dx$, that is, the upper curve minus the lower curve. If the graphs cross inside the interval, locate each crossing and treat every piece separately. Some regions are simpler in y : then take the right boundary minus the left boundary, $A = \int_c^d (x_{\text{right}} - x_{\text{left}}) dy$.

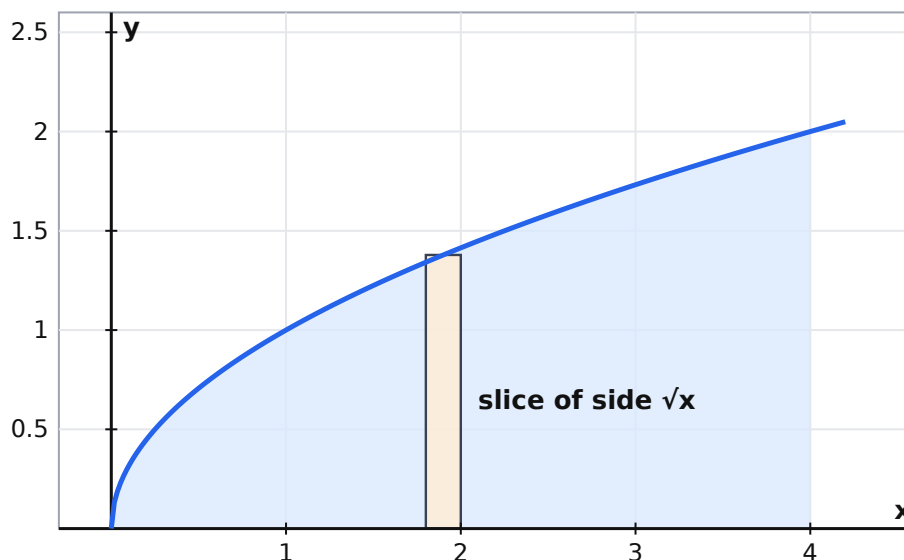


Area between curves: $\int (\text{top} - \text{bottom}) dx = \int_0^2 (2x - x^2) dx = 4/3$.

Worked case. The graphs of $y = 2x$ and $y = x^2$ cross at $x = 0$ and $x = 2$, and the line lies above the parabola between them. The enclosed area is $\int_0^2 (2x - x^2) dx = \left[x^2 - \frac{x^3}{3} \right]_0^2 = 4 - \frac{8}{3} = \frac{4}{3}$.

How are volumes built from cross sections?

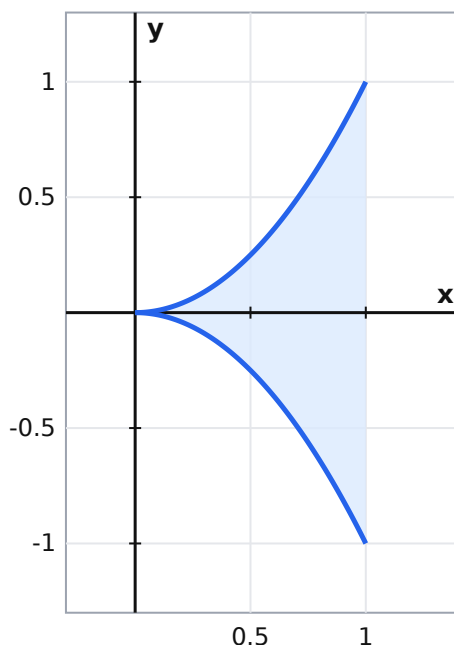
Slice a solid perpendicular to the x -axis. If a slice at position x has area $A(x)$, the solid has volume $V = \int_a^b A(x) dx$, the sum of all the slice areas. Useful shape formulas: a square with side s has area s^2 ; a semicircle on diameter d has area $\frac{\pi}{8}d^2$; an equilateral triangle with side s has area $\frac{\sqrt{3}}{4}s^2$. Express the side or diameter in terms of x first.



A square cross section at x has side $\sqrt{x}$, so its area is $A(x) = x$.

How do discs and washers work?

Spin a region around the x -axis and cut it into thin slices perpendicular to the axis. Every slice is a disc of area $\pi[f(x)]^2$, so $V = \pi \int_a^b [f(x)]^2 dx$. When a gap separates the region from the axis (outer curve $y = f(x)$, inner curve $y = g(x)$), each slice is a washer: $V = \pi \int_a^b ([f(x)]^2 - [g(x)]^2) dx$. A single example: spinning $y = x^2$ over $[0,1]$ gives $\pi \int_0^1 x^4 dx = \frac{\pi}{5}$.



Revolving $y = x^2$ about the x -axis; each disc has radius x^2 and area πx^4 .

Questions about average value, area and volume

How do you find the average value of a function?

Compute the definite integral of f across the interval, then divide that area by the interval length $b - a$.

What is the difference between displacement and distance?

Displacement integrates velocity, so backward motion subtracts and the result can be zero. Distance integrates speed $|v|$, so every part of the trip adds.

How do you find the area between two curves?

Locate the intersection points, identify the upper curve on each piece, and integrate upper minus lower. A crossing means the interval must be split.

How do cross sections give volume?

Describe one slice by its area $A(x)$, using the shape (square, semicircle or triangle) and the distance between the boundary curves, and integrate $A(x)$ across the solid.

What is the washer method?

If the region does not touch the axis of revolution, every slice is a ring. Integrate π times the outer radius squared minus the inner radius squared.