

# AP Calculus AB - Concavity and the Second Derivative - Answer Key

Common Core AP CALC AB 5.7, AP CALC AB 5.8, AP CALC AB 5.9 · [maths-pdf.fr/en/](https://maths-pdf.fr/en/)

This answer key gives a full worked solution for every problem in the concavity worksheet. Compare each step, not only the final answer, because AP graders award credit for a correct setup and clear justification. Final answers are in bold. Redo any problem you missed before moving on to the test.

**AP Calculus AB** · Common Core State Standards: **AP CALC AB 5.7, AP CALC AB 5.8, AP CALC AB 5.9**

## Exercise 1 - Concavity of a cubic

$$f' = 3x^2 - 12x + 9 \text{ and } f'' = 6x - 12. \text{ Set } f'' = 0: x = 2.$$

$$\text{For } x < 2, f'' < 0; \text{ for } x > 2, f'' > 0. f(2) = 8 - 24 + 18 + 1 = 3.$$

Final answer: concave down on  $(-\infty, 2)$ ; concave up on  $(2, \infty)$ ; inflection point  $(2, 3)$ .

## Exercise 2 - Concavity of a logarithm

$$f' = \frac{2x}{x^2+1} \text{ and } f'' = \frac{2(x^2+1) - 2x(2x)}{(x^2+1)^2} = \frac{2-2x^2}{(x^2+1)^2}.$$

The denominator is positive, so the sign is that of  $2(1-x)(1+x)$ : negative for  $|x| > 1$  and positive for  $|x| < 1$ .  $f(\pm 1) = \ln 2$ .

Final answer: concave up on  $(-1, 1)$ ; concave down on  $(-\infty, -1)$  and  $(1, \infty)$ ; inflections  $(\pm 1, \ln 2)$ .

## Exercise 3 - Exponential inflection

$$f' = e^{-x}(1-x) \text{ and } f'' = e^{-x}(x-2).$$

Zero at  $x = 2$ , changing from negative to positive.  $f(2) = 2e^{-2}$ .

Final answer:  $(2, 2e^{-2})$ .

### Exercise 4 - Always concave down

$f'' = -\frac{1}{x^2} < 0$  for all  $x > 0$ , so the graph never changes concavity.

Final answer: concave down on  $(0, \infty)$ ; no inflection point.

### Exercise 5 - Sine on an interval

$f'' = -\sin x$ . It is negative on  $(0, \pi)$  and positive on  $(\pi, 2\pi)$ .

Final answer: concave down on  $(0, \pi)$ ; concave up on  $(\pi, 2\pi)$ ; inflection point  $(\pi, 0)$ .

### Exercise 6 - Inflection where $f''$ is undefined

$f'' = -\frac{2}{9}x^{-5/3}$ . For  $x < 0$ ,  $x^{-5/3} < 0$  so  $f'' > 0$ ; for  $x > 0$ ,  $f'' < 0$ .

Final answer: concavity changes at  $x = 0$ : inflection point  $(0, 0)$ .

### Exercise 7 - The bell curve

$$f' = -\frac{2x}{(x^2+1)^2} \text{ and } f'' = \frac{6x^2-2}{(x^2+1)^3}.$$

Set  $6x^2 - 2 = 0$ :  $x = \pm \frac{1}{\sqrt{3}}$ . The sign changes at both.  $f\left(\pm \frac{1}{\sqrt{3}}\right) = \frac{3}{4}$ .

Final answer:  $\left(\pm \frac{1}{\sqrt{3}}, \frac{3}{4}\right)$ .

### Exercise 8 - Zero but not an inflection

$f'' = 12x^2$  is zero at 0 but positive on both sides.

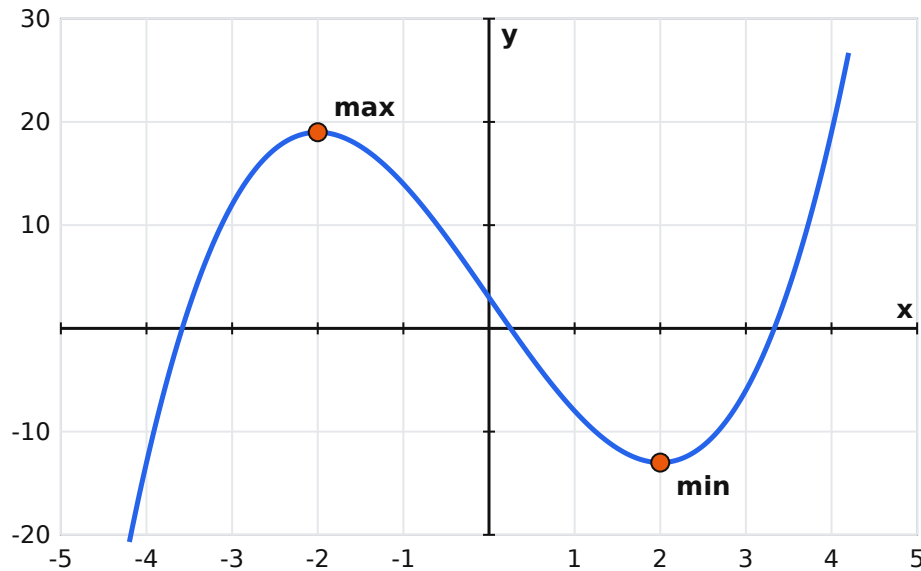
Final answer: No:  $f''$  does not change sign.

### Exercise 9 - Second derivative test, cubic

$$f' = 3x^2 - 12 = 0 \text{ gives } x = \pm 2. \quad f'' = 6x.$$

$f''(2) = 12 > 0$ : relative minimum.  $f''(-2) = -12 < 0$ : relative maximum.

Final answer:  $\max f(-2)=19$ ;  $\min f(2)=-13$ .



Maximum at  $x = -2$ , minimum at  $x = 2$ .

### Exercise 10 - Second derivative test, exponential

$f' = e^{-x}x(2-x)$ : critical numbers 0 and 2.  $f'' = e^{-x}(x^2 - 4x + 2)$ .

$f''(0) = 2 > 0$ : minimum.  $f''(2) = e^{-2}(4 - 8 + 2) = -2e^{-2} < 0$ : maximum.

Final answer:  $\min f(0) = 0$ ;  $\max f(2) = 4e^{-2}$ .

### Exercise 11 - A test that fails

$f' = 4x^3$  is zero at 0, and  $f''(0) = 12(0) = 0$ : inconclusive.

First Derivative Test:  $f' < 0$  for  $x < 0$  and  $f' > 0$  for  $x > 0$ .

Final answer: relative (and absolute) minimum  $f(0) = 0$ .

### Exercise 12 - Concavity from the graph of $f'$ prime

$f$  is concave up where  $f'$  is increasing.  $f'' = 3x^2 - 3 = 3(x-1)(x+1)$  is positive for  $|x| > 1$ .

The graph of  $f'$  has a local maximum at  $x = -1$  and a local minimum at  $x = 1$ :  $f''$  changes sign there.

Final answer: concave up on  $(-\infty, -1)$  and  $(1, \infty)$ ; inflection points at  $x = -1$  and  $x = 1$ .

### Exercise 13 - Conclusions from a sign table

Concave down where  $f'' < 0$ , concave up where  $f'' > 0$ . The sign changes at  $x = 0$  and  $x = 3$ .

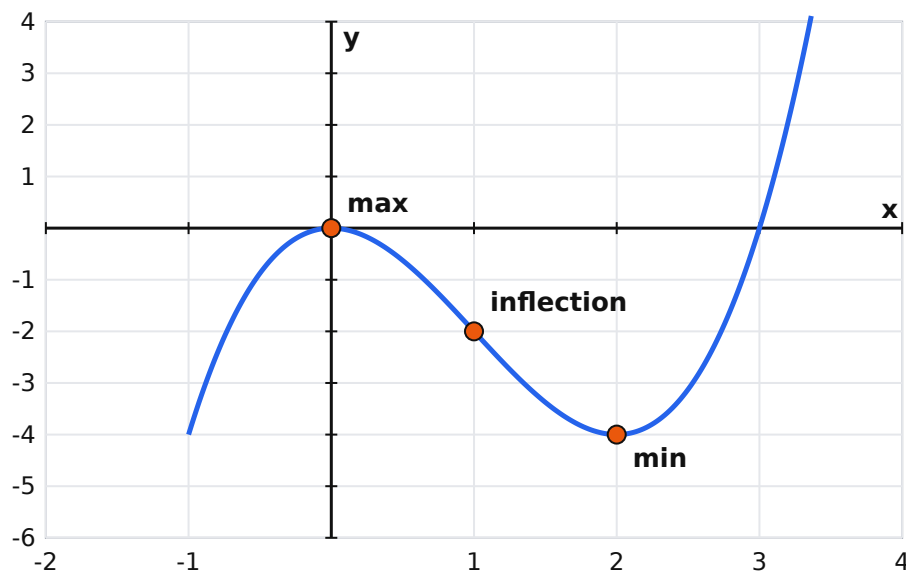
Final answer: down on  $(-\infty, 0)$  and  $(3, \infty)$ ; up on  $(0, 3)$ ; inflections at 0 and 3.

### Exercise 14 - All features of a cubic

$f' = 3x(x-2)$ : increasing on  $(-\infty, 0)$  and  $(2, \infty)$ , decreasing on  $(0, 2)$ . Relative max  $f(0) = 0$ , relative min  $f(2) = -4$ .

$f'' = 6x - 6$ : concave down on  $(-\infty, 1)$ , concave up on  $(1, \infty)$ , inflection point  $(1, -2)$ .

Final answer: max at  $(0, 0)$ , min at  $(2, -4)$ , inflection at  $(1, -2)$ .



Maximum, minimum and inflection point.

### Exercise 15 - Curve sketch

$f = x(x-3)^2$ , so the zeros are  $x = 0$  and  $x = 3$ .  $f' = 3(x-1)(x-3)$ ,  $f'' = 6x - 12$ .

$f''(1) = -6 < 0$ : max  $f(1) = 4$ .  $f''(3) = 6 > 0$ : min  $f(3) = 0$ . Inflection at  $x = 2$ ,  $f(2) = 2$ .

Final answer: max  $(1, 4)$ ; min  $(3, 0)$ ; inflection  $(2, 2)$ .

### Exercise 16 - Finding coefficients

$f'' = 6x + 2a$  is zero at  $x = 2$ :  $12 + 2a = 0$ , so  $a = -6$ .

$f'(1) = 3 + 2a + b = 0$  gives  $3 - 12 + b = 0$ , so  $b = 9$ .

Final answer:  $a = -6$ ,  $b = 9$ .

### Exercise 17 - Growth of sales

$S' = -3t^2 + 18t + 20$  and  $S'' = -6t + 18$ .  $S'' = 0$  at  $t = 3$ , changing from positive to negative.

$S'(3) = -27 + 54 + 20 = 47$ .

Final answer:  $t = 3$  months; rate 47 thousand dollars per month.

### Exercise 18 - Logistic population

Growth is fastest at the inflection point. At that time  $9e^{-t/2} = 1$ , so  $e^{-t/2} = \frac{1}{9}$  and  $t = 2\ln 9 \approx 4.39$ .

Then  $P = \frac{1000}{1+1} = 500$ .

Final answer:  $t \approx 4.39$  years, population 500.

### Exercise 19 - Marginal cost

$C' = 0.03x^2 - 1.8x + 30$  and  $C'' = 0.06x - 1.8$ , zero at  $x = 30$  and changing from negative to positive.

$C'(30) = 27 - 54 + 30 = 3$ .

Final answer:  $x = 30$  mugs, minimum marginal cost \$3.

### Exercise 20 - Both tests together

$f' = 5x^3(x-4)$ : critical numbers 0 and 4.  $f'' = 20x^3 - 60x^2 = 20x^2(x-3)$ .

$f''(4) = 320 > 0$ : relative minimum.  $f''(0) = 0$ : inconclusive; use signs of  $f'$ : + for  $x < 0$  and - for  $0 < x < 4$ .

Final answer: relative max at  $x=0$  ( $f=0$ ); relative min at  $x=4$  ( $f=-256$ ).

### Exercise 21 - Justifying inflection points

$f''=0$  at  $x=1$  and  $x=-2$ .

At  $x=1$ ,  $f''$  changes from negative to positive. At  $x=-2$  the factor  $(x+2)^2$  does not change sign ( $f''<0$  on both sides).

Final answer: *inflection point only at  $x=1$ .*

