

AP Calculus AB - The Derivative as a Limit - Answer Key

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This answer key gives a full worked solution for every problem in the derivative definition worksheet. Compare each step, not only the final answer, because AP graders award credit for a correct setup and clear justification. Final answers are in bold. Redo any problem you missed before moving on to the test.

AP Calculus AB · Common Core State Standards: **AP CALC AB 2.1, AP CALC AB 2.2, AP CALC AB 2.3, AP CALC AB 2.4**

Exercise 1 - Average rate of change

$f(1)=2$ and $f(4)=17$. The average rate is $\frac{17-2}{4-1}=\frac{15}{3}=5$.

Final answer: 5.

Exercise 2 - Secant slope from a table

$\frac{9.5-7}{0.5}=5$ and $\frac{13-9.5}{0.5}=7$.

Final answer: 5 and 7; the interval $[2.5,3]$ has the greater average rate.

Exercise 3 - Derivative at a point, x squared

$$\frac{(3+h)^2-9}{h}=\frac{6h+h^2}{h}=6+h.$$

As $h \rightarrow 0$, $6+h \rightarrow 6$. Final answer: $f'(3)=6$.

Exercise 4 - Derivative at a point, a quadratic

$f(1)=1$ and $f(1+h)=3(1+2h+h^2)-2-2h=1+4h+3h^2$.

$\frac{4h+3h^2}{h}=4+3h \rightarrow 4$. Final answer: $f'(1)=4$.

Exercise 5 - Derivative of a reciprocal

$$\frac{\frac{1}{2+h}-\frac{1}{2}}{h} = \frac{2-(2+h)}{2(2+h)h} = \frac{-h}{2(2+h)h} = \frac{-1}{2(2+h)}.$$

As $h \rightarrow 0$: $\frac{-1}{2 \cdot 2} = -\frac{1}{4}$. Final answer: $f'(2) = -\frac{1}{4}$.

Exercise 6 - Derivative of a square root

$$\frac{\sqrt{4+h}-2}{h} \text{ gives } \frac{0}{0}. \text{ Multiply by the conjugate: } \frac{(4+h)-4}{h(\sqrt{4+h}+2)} = \frac{1}{\sqrt{4+h}+2}.$$

As $h \rightarrow 0$: $\frac{1}{2+2} = \frac{1}{4}$. Final answer: $f'(4) = \frac{1}{4}$.

Exercise 7 - Derivative of a cube

$$(2+h)^3 = 8+12h+6h^2+h^3, \text{ so } \frac{(2+h)^3-8}{h} = 12+6h+h^2.$$

As $h \rightarrow 0$ this approaches 12. Final answer: $f'(2) = 12$.

Exercise 8 - The x-approaches-a form

$$f(3) = -6. \quad \frac{x^2-5x+6}{x-3} = \frac{(x-2)(x-3)}{x-3} = x-2 \text{ for } x \neq 3.$$

As $x \rightarrow 3$: $3-2=1$. Final answer: $f'(3) = 1$.

Exercise 9 - Derivative function and its zeros

$$f(x+h)-f(x) = 3(2xh+h^2)-(3x^2h+3xh^2+h^3) = 6xh-3x^2h+3h^2-3xh^2-h^3.$$

Divide by h : $6x-3x^2+3h-3xh-h^2$. As $h \rightarrow 0$, $f'(x) = 6x-3x^2 = 3x(2-x)$.

Final answer: $f'(x) = 6x-3x^2$; horizontal at $x=0$ and $x=2$.

Exercise 10 - Using the derivative function

$$f(x+h)-f(x) = 2(2xh+h^2)-h = 4xh+2h^2-h. \text{ Divide by } h: 4x+2h-1.$$

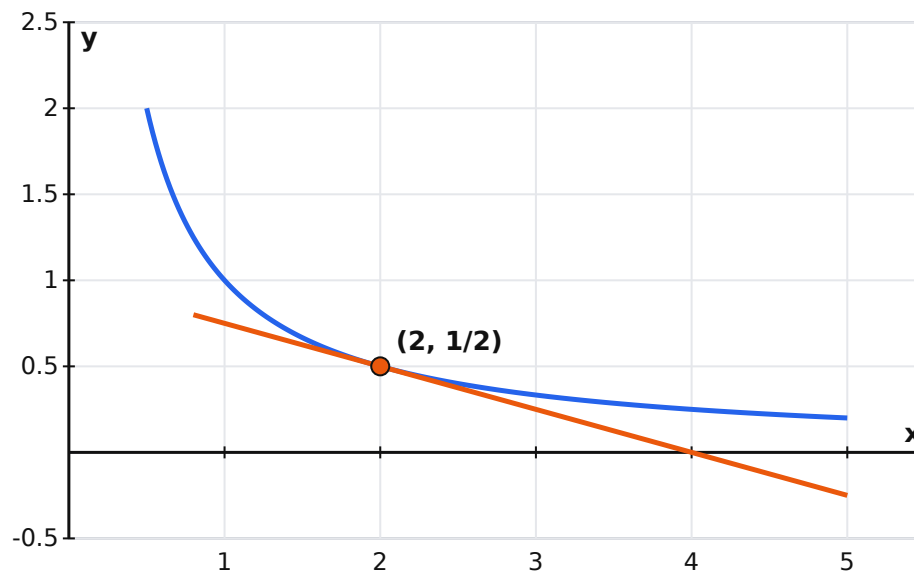
So $f'(x) = 4x-1$ and $f'(-2) = 4(-2)-1 = -9$. Final answer: $f'(x) = 4x-1$; $f'(-2) = -9$.

Exercise 11 - Tangent line to a reciprocal

$$f(2) = \frac{1}{2}. \quad \frac{f(2+h) - f(2)}{h} = \frac{\frac{1}{2+h} - \frac{1}{2}}{h} = \frac{-1}{2(2+h)} \rightarrow -\frac{1}{4}.$$

Point-slope form: $y - \frac{1}{2} = -\frac{1}{4}(x - 2).$

Final answer: $y = -\frac{1}{4}x + 1.$

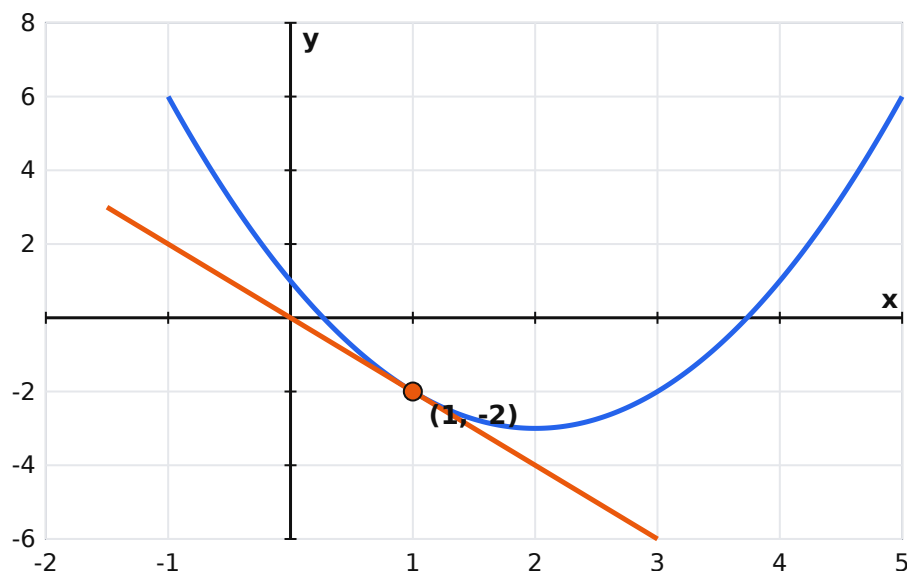


The tangent to $1/x$ at $x = 2$.

Exercise 12 - Tangent line, second example

$$f(1) = -2. \quad f(1+h) - f(1) = (1+2h+h^2) - 4 - 4h + 1 - (-2) = -2h + h^2, \text{ so the quotient is } -2+h \rightarrow -2.$$

Tangent: $y + 2 = -2(x - 1)$, so $y = -2x$. Final answer: $y = -2x$.



The tangent line $y = -2x$ at $(1, -2)$.

Exercise 13 - Slope from a graph

The derivative equals the slope of the tangent: $\frac{3-(-1)}{4-0} = 1$.

The tangent line is $y-1=1(x-2)$, which is $y = x-1$.

Final answer: $f'(2)=1$ and $y = x-1$.

Exercise 14 - Estimating from a table

$$\frac{f(2.1)-f(1.9)}{2.1-1.9} = \frac{7.161-4.959}{0.2} = 11.01.$$

Final answer: $f'(2) \approx 11.01$.

Exercise 15 - A corner

For $h > 0$: $\frac{|h|-0}{h} = 1$. For $h < 0$: $\frac{|h|-0}{h} = -1$.

The one-sided limits 1 and -1 differ, so **f is not differentiable at $x = 1$** (even though it is continuous there).

Exercise 16 - A vertical tangent

$$\frac{f(0+h)-f(0)}{h} = \frac{h^{1/3}}{h} = h^{-2/3} = \frac{1}{h^{2/3}}.$$

As $h \rightarrow 0$, $\frac{1}{h^{2/3}} \rightarrow \infty$. The tangent line is vertical, so the slope is undefined.

Final answer: **not differentiable at $x = 0$** .

Exercise 17 - Continuous but not differentiable

Continuity: left limit 1, right limit $3-2=1$, $f(1)=1$. So **continuous**.

Slopes: the left piece has derivative $2x=2$ at 1; the right piece has slope 3. They differ.

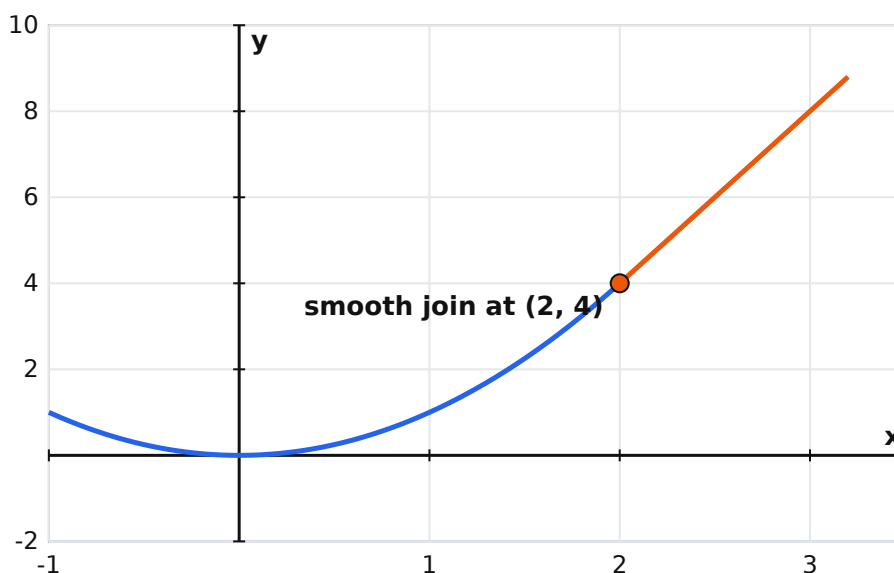
Final answer: continuous but **not differentiable** at $x=1$ (a corner).

Exercise 18 - Making a function differentiable

Slopes must match: the derivative of x^2 at 2 is 4, so $a=4$.

Values must match: $2a+b=4$, so $8+b=4$ and $b=-4$.

Final answer: $a=4$, $b=-4$.



With $a = 4$ and $b = -4$ the pieces join smoothly.

Exercise 19 - Speed from the definition

$$s(3)=36 \text{ and } s(3+h)=4(9+6h+h^2)=36+24h+4h^2.$$

$$\frac{24h+4h^2}{h} = 24+4h \rightarrow 24. \text{ Final answer: } 24 \text{ ft/s.}$$

Exercise 20 - Marginal cost

$$C(10) = 50 + 200 = 250. \quad C(10+h) = 0.5(100+20h+h^2) + 200 + 20h = 250 + 30h + 0.5h^2.$$

$$\frac{30h+0.5h^2}{h} = 30+0.5h \rightarrow 30.$$

Final answer: $C'(10) = 30$: at 10 shirts, each additional shirt costs about \\$30.

Exercise 21 - Absolute value at zero

$\frac{|h|-0}{h}$ equals 1 for $h > 0$ and -1 for $h < 0$. The one-sided limits differ, so $f'(0)$ does not exist.

For $x > 0$, $f(x) = x$ so $f'(x) = 1$. For $x < 0$, $f(x) = -x$ so $f'(x) = -1$.

Final answer: $f'(0)$ **does not exist**; $f' = 1$ for $x > 0$ and $f' = -1$ for $x < 0$.

Exercise 22 - Recognizing a derivative

a. This is $f'(2)$ for $f(x) = x^5$, and $f'(x) = 5x^4$, so the limit is $5 \cdot 16 = 80$.

b. This is $f'(e)$ for $f(x) = \ln x$ since $\ln e = 1$. With $f'(x) = \frac{1}{x}$, the limit is $\frac{1}{e}$.