

AP Calculus AB - Derivatives of Inverse Functions - Lesson

Common Core AP CALC AB 3.3, AP CALC AB 3.4, AP CALC AB 3.5, AP CALC AB 3.6 · maths-pdf.fr/en/

The derivative of an inverse function is the reciprocal of the original derivative: $(f^{-1})'(b) = 1/f'(a)$, where $f(a) = b$. The derivatives of the inverse trig functions are $1/\sqrt{1-x^2}$ for $\sin^{-1}x$, $-1/\sqrt{1-x^2}$ for $\cos^{-1}x$ and $1/(1+x^2)$ for $\tan^{-1}x$. Higher-order derivatives repeat differentiation.

AP Calculus AB · Common Core State Standards: **AP CALC AB 3.3, AP CALC AB 3.4, AP CALC AB 3.5, AP CALC AB 3.6**

Key facts

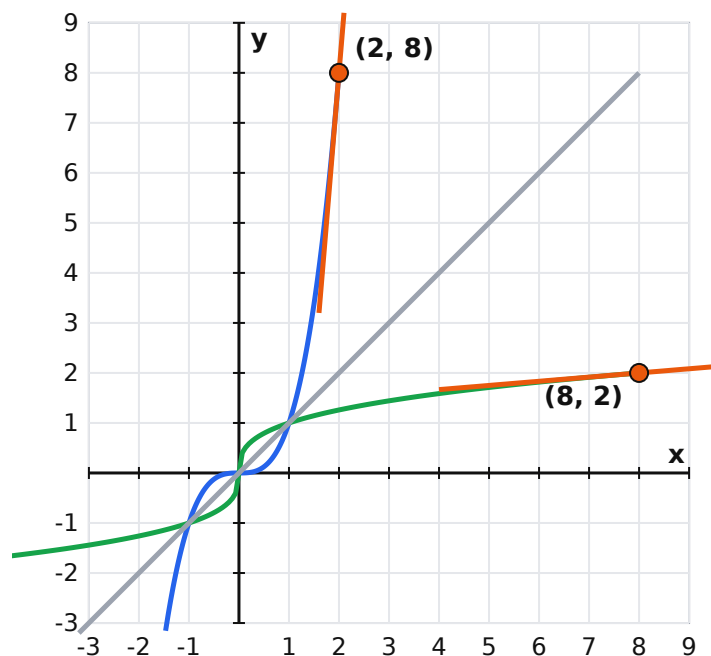
- $(f^{-1})'(b) = 1/f'(a)$ where $f(a) = b$; evaluate f' at a , not at b .
- $d/dx \sin^{-1}x = 1/\sqrt{1-x^2}$, $d/dx \cos^{-1}x = -1/\sqrt{1-x^2}$, $d/dx \tan^{-1}x = 1/(1+x^2)$.
- The second derivative is the derivative of the first derivative; acceleration is the second derivative of position.
- Choose the rule by the outermost operation: sum, product, quotient or composition.

How do you differentiate an inverse function?

If f has an inverse f^{-1} , then the graphs are mirror images across the line $y = x$. Reflection swaps the roles of rise and run, so the slopes are reciprocals.

Derivative of an inverse

$(f^{-1})'(b) = \frac{1}{f'(a)}$ where $f(a) = b$, provided $f'(a) \neq 0$. Evaluate f' at the *input of the original function*.



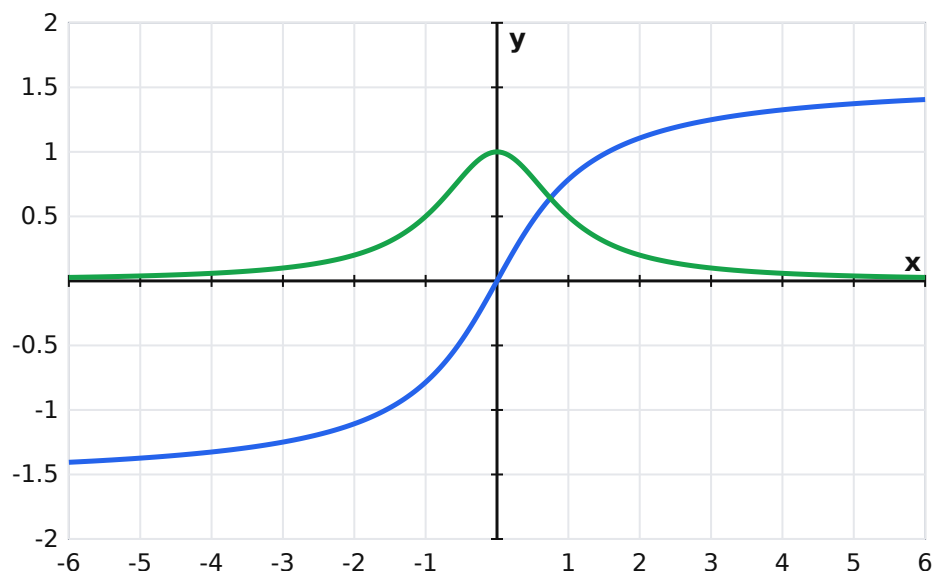
Reflecting across $y = x$ swaps the points $(2, 8)$ and $(8, 2)$ and turns the slope 12 into $1/12$.

Example. For $f(x) = x^3$, $f(2) = 8$ and $f'(2) = 12$. So $(f^{-1})'(8) = \frac{1}{12}$.

What are the derivatives of inverse trig functions?

Function	Derivative
$\sin^{-1}x$	$\frac{1}{\sqrt{1-x^2}}$
$\cos^{-1}x$	$-\frac{1}{\sqrt{1-x^2}}$
$\tan^{-1}x$	$\frac{1}{1+x^2}$

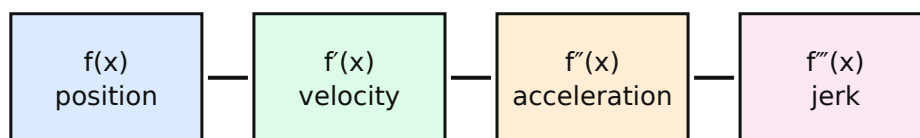
When the input is more than x , add the chain rule. For $\tan^{-1}(x^2)$, the derivative is $\frac{2x}{1+x^4}$.



The derivative $1/(1 + x^2)$ (green) is the slope of $\arctan x$ (blue).

What are higher-order derivatives?

The second derivative f'' is the derivative of f' . It is also written $\frac{d^2y}{dx^2}$. In motion problems, if $s(t)$ is position, then $v = s'$ is velocity and $a = s''$ is acceleration.



Each derivative is the rate of change of the one before it.

Example. For $f(x) = x^4 - 3x^3 + 2x$: $f' = 4x^3 - 9x^2 + 2$, $f'' = 12x^2 - 18x$ and $f''' = 24x - 18$.

Example with a table. If $f(1) = 7$ and $f'(1) = 3$, then the inverse passes through $(7, 1)$ with slope $\frac{1}{3}$. Be careful about the order: the input of f^{-1} is 7, but the derivative of f is evaluated at 1. Many errors come from using $f'(7)$ by mistake.

Higher-order derivatives also reveal patterns. For e^{3x} , each derivative multiplies by 3, so the fourth derivative is $81e^{3x}$. For $\sin x$, the derivatives repeat every four steps: $\cos x$, $-\sin x$, $-\cos x$, $\sin x$.

How do you choose a method?

Read the outermost operation first. A sum is differentiated term by term. A product needs the product rule, a quotient needs the quotient rule, and a function inside another function needs the chain rule. If y appears in an equation, use implicit differentiation.

You see	Use
A sum or difference	Term by term
$f(x) \cdot g(x)$	Product rule
$f(x) / g(x)$	Quotient rule
$f(g(x))$	Chain rule
x and y mixed	Implicit

A quick guide for choosing a differentiation procedure.

Questions about derivatives of inverse functions

How do you find the derivative of an inverse function?

Find the input a with $f(a) = b$, compute $f'(a)$, and take its reciprocal. The result is $(f \text{ inverse})'(b) = 1/f'(a)$.

Why are the slopes of f and its inverse reciprocals?

The graphs are reflections across $y = x$, which swaps rise and run. A tangent line with slope m becomes a tangent line with slope $1/m$.

What is the derivative of $\arctan x$?

The derivative of $\arctan x$ is $1/(1 + x^2)$. It is always positive, which matches the increasing graph of $\arctan$.

What is the second derivative?

The second derivative is the derivative of the first derivative. It measures how fast the slope is changing, and in motion it is the acceleration.

How do you decide which differentiation rule to use?

Look at the outermost operation. Sums use term-by-term differentiation, products and quotients use their own rules, nested functions use the chain rule, and equations mixing x and y use implicit differentiation.