

AP Calculus AB - Linearization and L'Hopital's Rule - Lesson

Common Core AP CALC AB 4.5, AP CALC AB 4.6, AP CALC AB 4.7 · maths-pdf.fr/en/

A local linear approximation uses the tangent line $L(x) = f(a) + f'(a)(x - a)$ to estimate f near $x = a$. It overestimates when the graph is concave down and underestimates when it is concave up. L'Hôpital's Rule evaluates limits of the form $0/0$ or ∞/∞ by differentiating the numerator and denominator separately.

AP Calculus AB · Common Core State Standards: **AP CALC AB 4.5, AP CALC AB 4.6, AP CALC AB 4.7**

Key facts

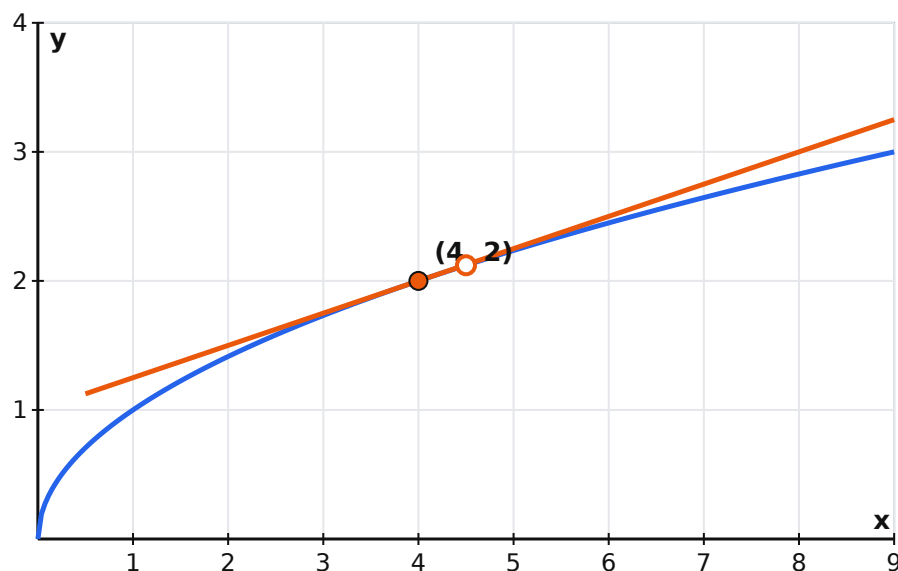
- Linearization: $L(x) = f(a) + f'(a)(x - a)$, the tangent line at $x = a$.
- Concave up (f'' positive): the tangent line is below the curve, so the estimate is too low.
- Concave down (f'' negative): the tangent line is above the curve, so the estimate is too high.
- L'Hôpital: for $0/0$ or ∞/∞ , $\lim f/g = \lim f'/g'$ (not the quotient rule).

What is a local linear approximation?

Near a point $x = a$, a differentiable curve is almost a straight line: its tangent line. The tangent line gives a simple formula that approximates the function for inputs close to a .

Linearization

$L(x) = f(a) + f'(a)(x - a)$, and $f(x) \approx L(x)$ for x near a . Choose a so that $f(a)$ and $f'(a)$ are easy to compute.

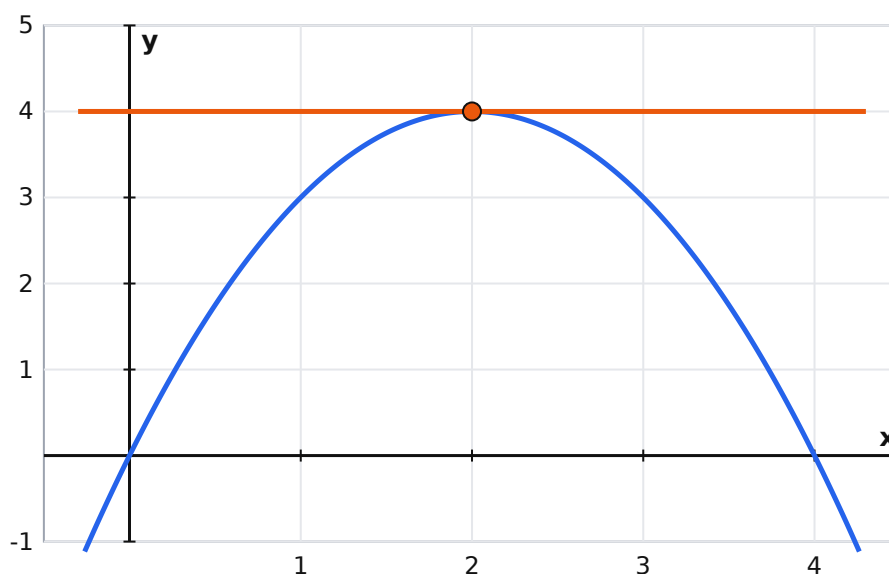


Near $x = 4$ the tangent line is almost the same as the curve.

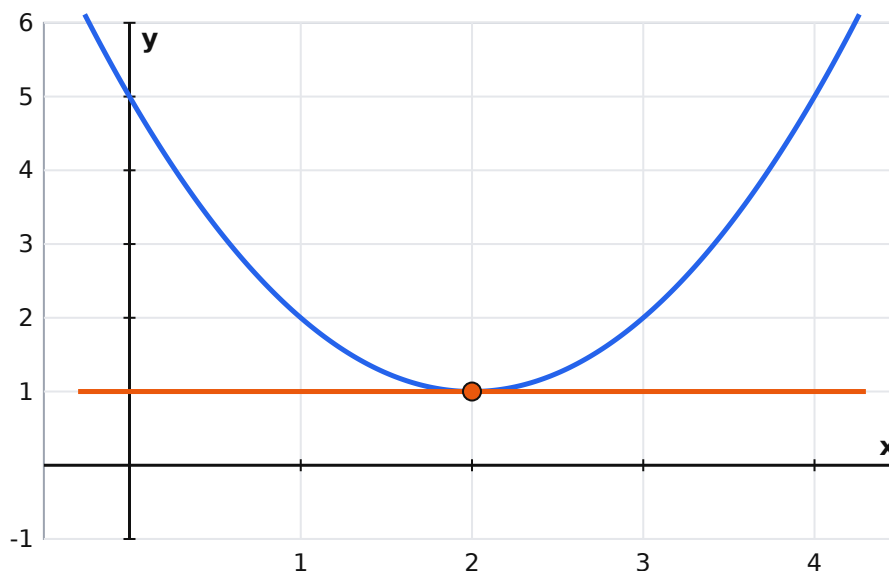
Example. Estimate $\sqrt{4.5}$. Take $f(x) = \sqrt{x}$, $a = 4$, $f(4) = 2$ and $f'(4) = \frac{1}{4}$. Then $L(x) = 2 + \frac{1}{4}(x-4)$ and $L(4.5) = 2.125$. The calculator value is 2.1213, so the estimate is slightly high.

When is the estimate too high or too low?

Check the concavity. If $f'' > 0$ (concave up) near a , the tangent line lies *below* the curve, so $L(x)$ is an **underestimate**. If $f'' < 0$ (concave down), the tangent line lies *above*, so it is an **overestimate**. Square root is concave down, which explains the example.



Concave down: the tangent lies above the curve (overestimate).



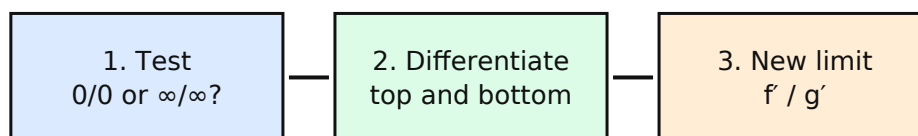
Concave up: the tangent lies below the curve (underestimate).

The approximation gets better as x gets closer to a . For $\sqrt{4.5}$ the error is about 0.004, but using the same tangent line to estimate $\sqrt{9}$ gives $2 + \frac{5}{4} = 3.25$, which is far from the true value 3. Always stay close to the point of tangency, and choose a point where the function value is a nice number such as a perfect square or cube.

What is L'Hôpital's Rule?

L'Hôpital's Rule

If $\lim_{x \rightarrow a} \frac{f(x)}{g(x)}$ has the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$, then $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$ provided the second limit exists (or is infinite).



Use L'Hôpital's Rule only on the indeterminate forms $0/0$ and ∞/∞ .

Example. $\lim_{x \rightarrow 0} \frac{e^x - 1}{x}$ is $\frac{0}{0}$. Differentiate top and bottom: $\lim_{x \rightarrow 0} \frac{e^x}{1} = 1$. Another: $\lim_{x \rightarrow \infty} \frac{x}{e^x}$ is $\frac{\infty}{\infty}$, and $\lim_{x \rightarrow 0} \frac{1}{e^x} = 0$.

Differentiate the numerator and denominator *separately*; do not use the quotient rule. If the new limit is still indeterminate, apply the rule again. Never use the rule when the form is not indeterminate: $\lim_{x \rightarrow 0} \frac{x+2}{x-1} = -2$ by substitution.

Questions about linearization and l'hopital's rule

What is a linear approximation?

It is the tangent line used to estimate a function near the point of tangency. The formula is $L(x) = f(a) + f'(a)(x - a)$.

How do you tell if a linear approximation is an overestimate?

Check the concavity. If the function is concave down near a , the tangent line lies above the graph, so the approximation is an overestimate. Concave up gives an underestimate.

When can you use L'Hôpital's Rule?

Only when substituting gives the indeterminate form $0/0$ or infinity over infinity. If the form is anything else, evaluate the limit another way.

Is L'Hôpital's Rule the same as the quotient rule?

No. L'Hôpital differentiates the numerator and denominator separately and then takes a limit. The quotient rule differentiates the whole fraction.

What if the limit is still $0/0$ after one application?

Apply the rule again. Keep differentiating the numerator and denominator until the form is no longer indeterminate.