

AP Calculus AB - Riemann Sums and the Definite Integral - Lesson

Common Core AP CALC AB 6.1, AP CALC AB 6.2, AP CALC AB 6.3 · maths-pdf.fr/en/

A Riemann sum approximates the area under a curve, and the accumulated change of a rate, by adding the areas of thin rectangles. Left, right and midpoint sums use different sample points, and the trapezoidal sum averages left and right. The definite integral is the limit of Riemann sums as the number of rectangles grows without bound.

AP Calculus AB · Common Core State Standards: **AP CALC AB 6.1, AP CALC AB 6.2, AP CALC AB 6.3**

Key facts

- Riemann sum: add $f(\text{sample point})$ times the width over all subintervals.
- For an increasing function a left sum underestimates and a right sum overestimates; the reverse for a decreasing function.
- The trapezoidal sum equals the average of the left and right sums when widths are equal.
- The definite integral is the limit of Riemann sums and gives signed area.

What is accumulation of change?

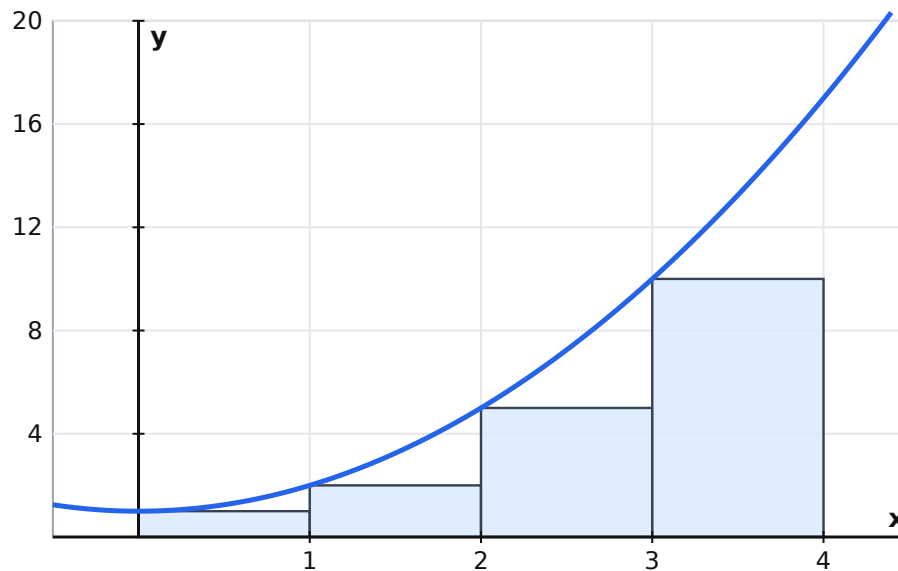
If a rate of change is known, the total change is found by accumulating it. A car with speed $v(t)$ travels a distance equal to the area between the graph of v and the t -axis. When the rate is not constant, we approximate that area with thin rectangles.

What is a Riemann sum?

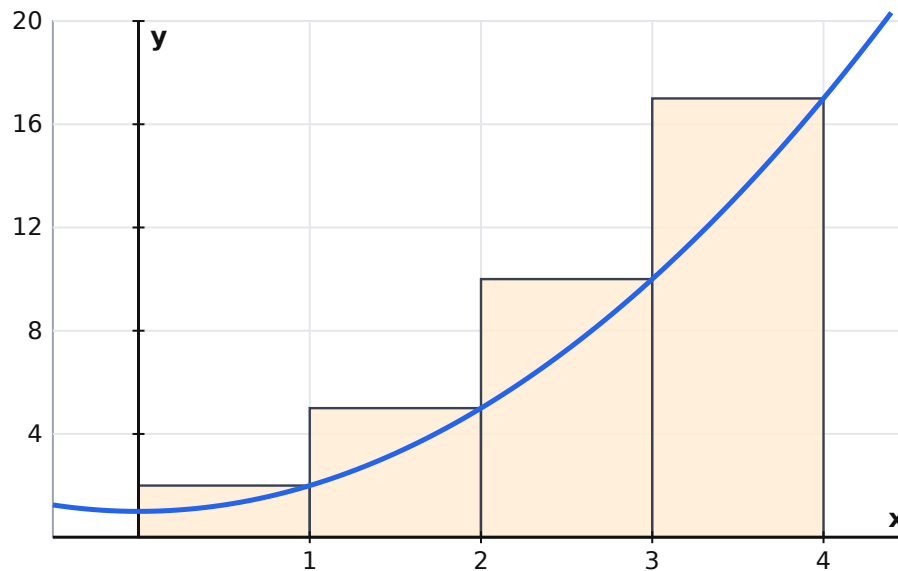
Divide $[a, b]$ into n subintervals of width $\Delta x = \frac{b-a}{n}$. On each subinterval build a rectangle whose height is the function value at a *sample point*. The sum of the rectangle areas is a **Riemann sum**.

Three standard sample points

- **Left sum:** use the left endpoint of each subinterval.
- **Right sum:** use the right endpoint.
- **Midpoint sum:** use the middle of each subinterval.



Left sum with $n = 4$: heights 1, 2, 5, 10 give 18, an underestimate for an increasing function.

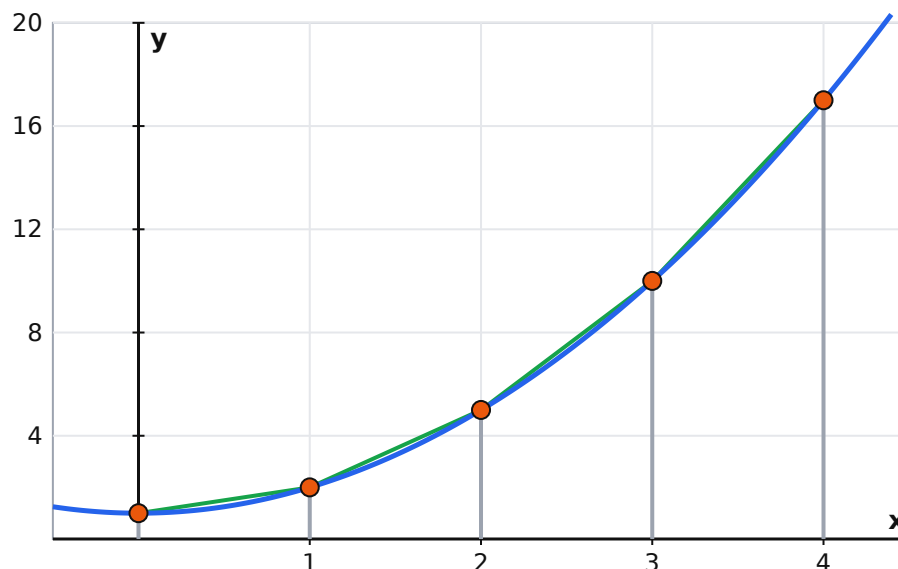


Right sum with $n = 4$: heights 2, 5, 10, 17 give 34, an overestimate for an increasing function.

Example. For $f(x) = x^2 + 1$ on $[0, 4]$ with $n = 4$, $\Delta x = 1$. The left sum is $1 \cdot (1 + 2 + 5 + 10) = 18$ and the right sum is $1 \cdot (2 + 5 + 10 + 17) = 34$. Because f is increasing, the left sum underestimates and the right sum overestimates the true area.

What is the trapezoidal sum?

A trapezoid with parallel sides $f(x_i)$ and $f(x_{i+1})$ has area $\frac{f(x_i) + f(x_{i+1})}{2} \Delta x$. Adding them equals the average of the left and right sums: $\frac{18 + 34}{2} = 26$. When the subintervals have different widths, use each width separately.



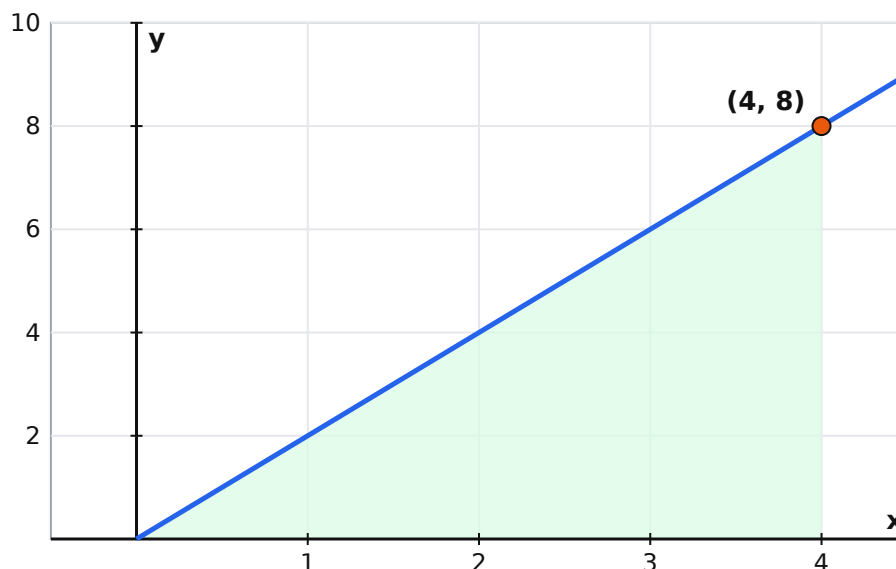
Trapezoidal sum: average of the left and right sums, $(18 + 34)/2 = 26$.

What is a definite integral?

Definite integral

$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*) \Delta x$ is the limit of Riemann sums. It gives the *signed* area between the graph and the x -axis: area above the axis counts positive, area below counts negative.

For simple shapes, use geometry: the integral $\int_0^4 2x dx$ is the area of a triangle with base 4 and height 8, so it equals 16. Useful properties: $\int_a^a f = 0$, $\int_b^a f = -\int_a^b f$, $\int_a^c f = \int_a^b f + \int_b^c f$ and $\int_a^b kf = k \int_a^b f$.



The integral of $2x$ from 0 to 4 equals the area of a triangle: $(1/2)(4)(8) = 16$.

Questions about riemann sums and the definite integral

What is a Riemann sum?

A Riemann sum is a sum of areas of rectangles that approximates the area under a curve. Each rectangle has width Δx and height equal to the function value at a sample point in the subinterval.

When is a left sum an underestimate?

When the function is increasing on the interval, each left rectangle lies below the curve, so the left sum underestimates. For a decreasing function it overestimates.

How do you find a trapezoidal sum?

Add the areas of the trapezoids: for each subinterval, take the average of the two endpoint function values and multiply by the width of that subinterval.

What does a definite integral represent?

It represents the signed area between the graph of f and the x -axis from a to b , and the net change of a quantity whose rate is f .

How can geometry find an integral?

When the graph is made of lines or circles, the signed area can be computed with area formulas for triangles, rectangles, trapezoids and circles, counting regions below the axis as negative.