

AP Calculus AB - Integration by Substitution - Lesson

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Integration by substitution reverses the chain rule. Choose u as the inner function whose derivative appears in the integrand, replace u and du , integrate in u , and substitute back. For definite integrals, convert the limits to u . When substitution alone is not enough, use long division or complete the square first.

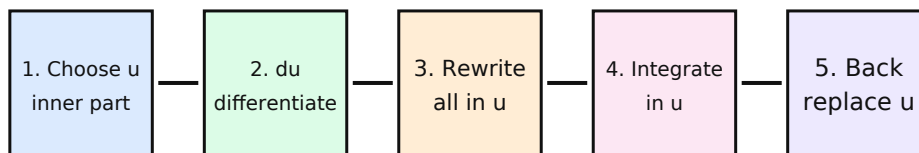
AP Calculus AB · Common Core State Standards: **AP CALC AB 6.9, AP CALC AB 6.10, AP CALC AB 6.11, AP CALC AB 6.12, AP CALC AB 6.14**

Key facts

- Let $u = g(x)$ and $du = g'(x) dx$, then the integral becomes the integral of $f(u) du$.
- Adjust constant factors: if du is off by a constant, divide or multiply to compensate.
- For definite integrals, change the limits to u -values or return to x before evaluating.
- Use long division for improper rational functions and completing the square for irreducible quadratics.

What is integration by substitution?

Substitution reverses the chain rule. If an integrand has the form $f(g(x)) g'(x)$, set $u = g(x)$ so that $du = g'(x) dx$, and the integral becomes the simpler $\int f(u) du$.



A five-step routine for integration by substitution.

Example. Find $\int 2x(x^2+1)^3 dx$. Let $u = x^2+1$, so $du = 2x dx$. Then $\int u^3 du = \frac{u^4}{4} + C = \frac{(x^2+1)^4}{4} + C$. Check by differentiating with the chain rule.

How do you choose u ?

Look for an expression whose derivative (up to a constant factor) also appears in the integrand. Good candidates are the inside of a power, the exponent of e , the argument of a

trig function, or the denominator.

Pattern	Substitution u	Result
$f(g(x)) g'(x)$	$g(x)$	$\int f(u) du$
$g'(x)/g(x)$	$g(x)$	$\ln u $
$e^{g(x)} g'(x)$	$g(x)$	e^u
$\sin x \cos^n x$	$\cos x$	$-u^{n+1}/(n+1)$

Patterns that signal a substitution.

If the derivative is off by a constant, fix it: for $\int x e^{x^2} dx$, $du = 2x dx$ gives $x dx = \frac{1}{2} du$, so the answer is $\frac{1}{2} e^{x^2} + C$.

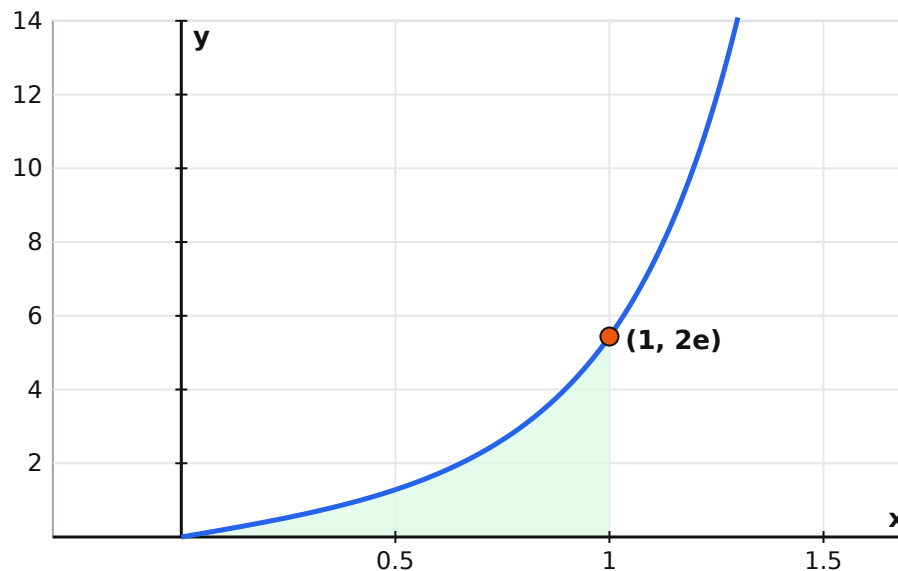
How do definite integrals change?

Either convert the limits to u and never go back to x , or find the antiderivative in x and use the original limits. Do not mix the two.

	x	$u = x^2 + 1$
lower limit	0	1
upper limit	2	5

When you substitute in a definite integral, convert the limits to u .

$$\int_0^2 2x(x^2+1)^3 dx = \int_1^5 u^3 du = \left[\frac{u^4}{4} \right]_1^5 = \frac{625-1}{4} = 156.$$



The shaded area equals $\int_0^1 2x e^{(x^2)} dx = e - 1 \approx 1.718$.

- **Check your answer** by differentiating. If the chain rule returns the original integrand, the antiderivative is right.
- **Keep the constant** $+C$ in every indefinite integral.
- **Never leave** x and u mixed in the same integral. Everything, including dx , must be written in u before you integrate.

For example, in $\int x\sqrt{x^2+4} dx$ with $u = x^2+4$, the factor $x dx$ is replaced by $\frac{1}{2}du$, giving $\frac{1}{2}\int u^{1/2}du = \frac{1}{3}u^{3/2}$. If a stray x remains after the substitution, either another choice of u is needed or the integral requires a different method.

What if substitution is not enough?

For a rational function whose numerator has degree at least the denominator, use **long division** first: $\frac{x^2+3x+1}{x+1} = x+2 - \frac{1}{x+1}$. For $\frac{1}{x^2+4x+13}$, **complete the square** to get $\frac{1}{(x+2)^2+9}$, and then use $\int \frac{1}{u^2+a^2}du = \frac{1}{a}\tan^{-1}\frac{u}{a}$.

Questions about integration by substitution

What is u-substitution?

It is a technique that rewrites an integral in terms of a new variable $u = g(x)$ so that the integrand becomes simpler. It undoes the chain rule.

How do you choose u ?

Pick the inner function (inside parentheses, a root, an exponent or a denominator) whose derivative, up to a constant, also appears in the integrand.

What happens to the limits of a definite integral?

If you rewrite in u , convert each limit with $u = g(x)$. Then evaluate in u without returning to x . Alternatively, find the antiderivative in x and use the original limits.

What if du does not match exactly?

A constant factor can be adjusted. For example, if $du = 2x \, dx$ but the integral has $x \, dx$, replace $x \, dx$ by $(1/2) \, du$.

When do you use long division in integration?

Use it when the numerator of a rational function has degree greater than or equal to the denominator. Divide first, then integrate each term.