

Precalculus - The Binomial Theorem - Lesson

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The Binomial Theorem expands $(a + b)^n$ as the sum of $C(n, k) a^{(n-k)} b^k$ for k from 0 to n , where $C(n, k) = n!/(k!(n - k)!)$ counts the ways to choose k items from n . The coefficients are the entries in row n of Pascal's triangle, and they add up to 2^n .

Precalculus · Common Core State Standards: **A-APR.C.5**

Key facts

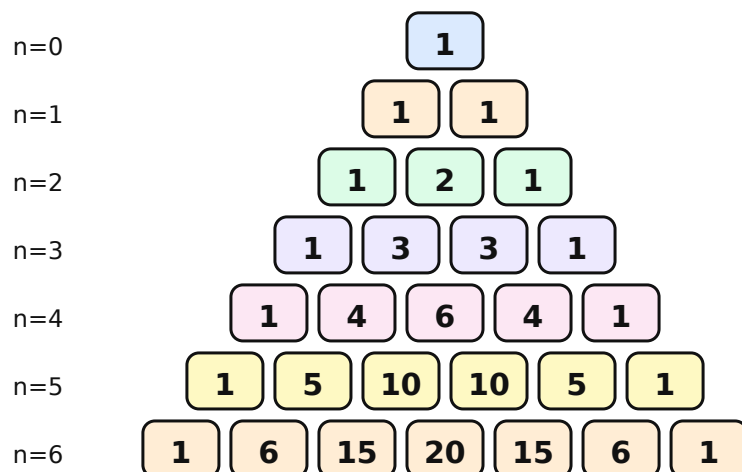
- $n! = n(n - 1) \dots (1)$ and $0! = 1$; $C(n, k) = n!/(k!(n - k)!)$.
- Pascal's triangle: each entry is the sum of the two above; row n sums to 2^n .
- $(a + b)^n = \text{sum of } C(n, k) a^{(n-k)} b^k \text{ for } k = 0 \text{ to } n$.
- The general term is $T(k+1) = C(n, k) a^{(n-k)} b^k$; signs alternate when b is negative.
- Setting $x = 1$ gives the sum of the coefficients.

What are factorials and binomial coefficients?

The factorial $n!$ is the product $n(n-1)(n-2) \dots (1)$, with $0! = 1$. The binomial coefficient $C(n, k) = \frac{n!}{k!(n-k)!}$ counts the ways to choose k items from n when order does not matter. For example $C(5, 2) = \frac{5!}{2!3!} = \frac{120}{12} = 10$. Two useful facts are $C(n, 0) = C(n, n) = 1$ and $C(n, k) = C(n, n-k)$.

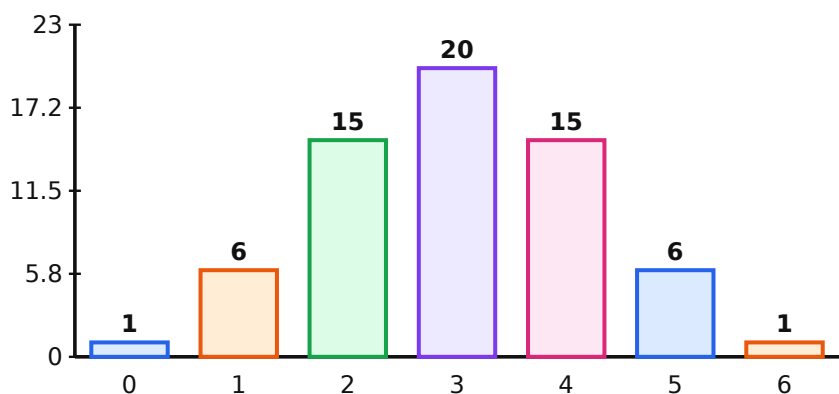
What is Pascal's triangle?

Pascal's triangle lists the binomial coefficients: row n contains $C(n, 0), C(n, 1), \dots, C(n, n)$. Each entry is the sum of the two entries above it, because $C(n, k) = C(n-1, k-1) + C(n-1, k)$. The entries in row n add up to 2^n .



Pascal's triangle. Row 5 (shaded) is 1, 5, 10, 10, 5, 1, and its sum is $32 = 2^5$.

Look at the shape of each row. The entries rise to the middle and fall again, and the row reads the same forward and backward, which is the symmetry $C(n,k) = C(n,n-k)$. This symmetry is useful: you only need to compute half of a row, and you can check an expansion by seeing that the coefficients of the first and last terms match. The rows also contain surprises, such as the powers of 11 (row 4 gives 1,4,6,4,1, and $11^4 = 14641$).



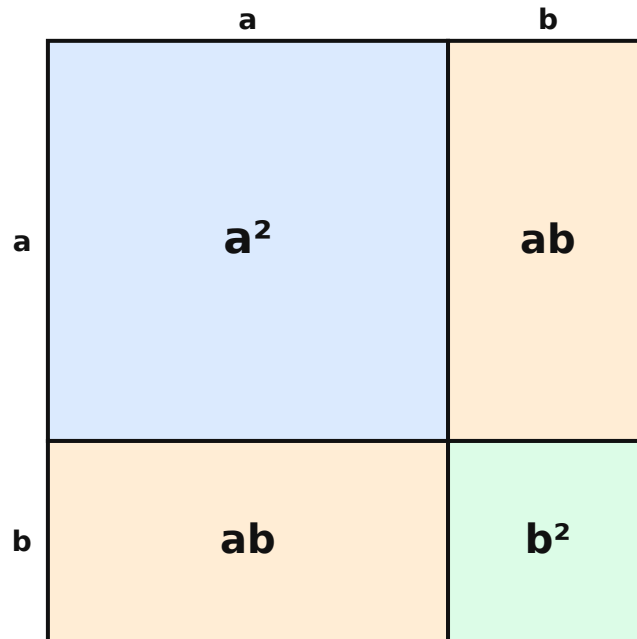
The entries of row 6 are symmetric: 1, 6, 15, 20, 15, 6, 1.

How do you expand a power of a binomial?

Binomial Theorem

$(a+b)^n = \sum_{k=0}^n C(n,k) a^{n-k} b^k$. The terms have exponents of a that decrease from n to 0 and exponents of b that increase from 0 to n .

Squaring shows the idea geometrically: $(a+b)^2 = a^2 + 2ab + b^2$, where the 2 is the coefficient $C(2,1)$.



A square with side $a + b$ splits into regions a^2 , ab , ab and b^2 .

Example: Expand $(x+2)^5$. Use row 5 of Pascal's triangle and powers of 2.

k	0	1	2	3	4	5
$C(5,k)$	1	5	10	10	5	1
$x^{(5-k)}$	x^5	x^4	x^3	x^2	x	1
2^k	1	2	4	8	16	32
term	$1x^5$	$10x^4$	$40x^3$	$80x^2$	$80x$	32

Each term is $C(5,k) \cdot x^{5-k} \cdot 2^k$. The sum is $x^5 + 10x^4 + 40x^3 + 80x^2 + 80x + 32$.

When the second term is negative, the signs alternate: $(x-3)^3 = x^3 - 9x^2 + 27x - 27$. To find a single term, use the general term $T_{k+1} = C(n,k) a^{n-k} b^k$ instead of expanding everything. The sum of all coefficients of $(ax+b)^n$ is found by setting $x=1$, giving $(a+b)^n$. Binomial coefficients also solve counting and probability problems: the number of ways to get exactly 3 heads in 5 tosses is $C(5,3) = 10$.

Questions about the binomial theorem

What does $C(n, k)$ mean?

It is the number of ways to choose k objects from n when order does not matter, read 'n choose k'. For example $C(5, 2) = 10$.

How does Pascal's triangle give the coefficients?

Row n lists $C(n, 0)$ through $C(n, n)$. Row 4 is 1, 4, 6, 4, 1, so $(a + b)^4 = a^4 + 4a^3 b + 6a^2 b^2 + 4a b^3 + b^4$.

How many terms are in the expansion of $(a + b)^n$?

There are $n + 1$ terms, one for each value of k from 0 to n .

How do I find one term without expanding?

Use the general term $C(n, k) a^{(n-k)} b^k$ with the k that gives the power you want. The $(k + 1)$ th term uses k .

Why do the signs alternate for $(x - 3)^n$?

Because b is negative, b^k is negative for odd k and positive for even k , so the terms alternate in sign.