

Precalculus - Inverse Functions - Lesson

Common Core F-BF.B.4a, F-BF.B.4b, F-BF.B.4c, F-BF.B.4d · maths-pdf.fr/en/

An inverse function f^{-1} undoes f : if $f(a) = b$, then $f^{-1}(b) = a$. Only one-to-one functions, which pass the horizontal line test, have inverses. To find an inverse, write $y = f(x)$, swap x and y , and solve for y . The graph of f^{-1} is the reflection of f over the line $y = x$.

Precalculus · Common Core State Standards: **F-BF.B.4a, F-BF.B.4b, F-BF.B.4c, F-BF.B.4d**

Key facts

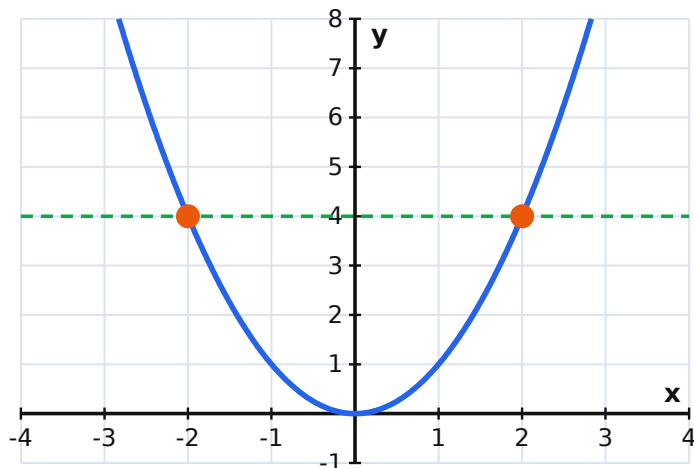
- $f^{-1}(f(x)) = x$ and $f(f^{-1}(x)) = x$.
- Find an inverse: write $y = f(x)$, swap x and y , solve for y .
- Domain of f becomes the range of f^{-1} , and the range of f becomes the domain of f^{-1} .
- The graph of f^{-1} is the reflection of the graph of f over $y = x$.
- Restrict the domain of a function that is not one-to-one, such as x^2 with $x \geq 0$.

What is an inverse function?

The inverse of f , written f^{-1} , undoes f . If $f(a) = b$, then $f^{-1}(b) = a$. That means $f^{-1}(f(x)) = x$ for every x in the domain of f , and $f(f^{-1}(x)) = x$ for every x in the domain of f^{-1} . The notation f^{-1} is not a reciprocal: $f^{-1}(x) \neq \frac{1}{f(x)}$.

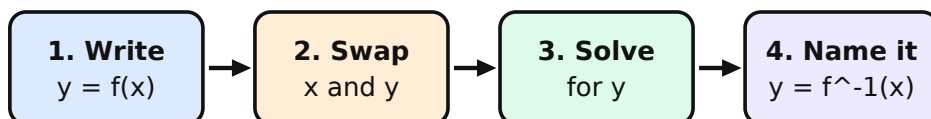
One-to-one

A function is **one-to-one** if each output comes from exactly one input. Only one-to-one functions have inverse functions. The **horizontal line test**: no horizontal line meets the graph more than once.



A parabola fails the horizontal line test: $y = 4$ hits it twice.

How do you find an inverse algebraically?



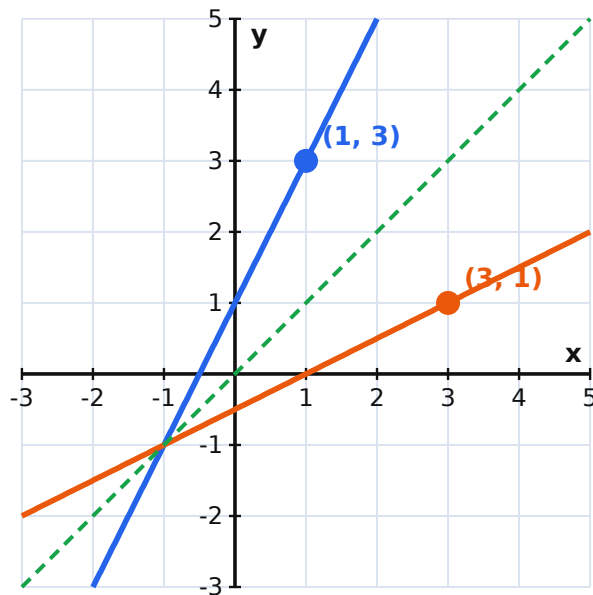
Four steps for finding f inverse.

Example: $f(x) = 2x + 1$. Write $y = 2x + 1$. Swap: $x = 2y + 1$. Solve: $y = \frac{x-1}{2}$. So $f^{-1}(x) = \frac{x-1}{2}$.
Check: $f(f^{-1}(x)) = 2 \cdot \frac{x-1}{2} + 1 = x$.

Why does this work? The rule f multiplies by 2 and then adds 1. To undo it, reverse the steps in the opposite order: subtract 1, then divide by 2. Swapping x and y and solving does exactly that automatically, even for rules with several steps. When the rule has a square, a root or a fraction, check the domain at the end, because the inverse may be valid only for some inputs.

How are the graphs of f and its inverse related?

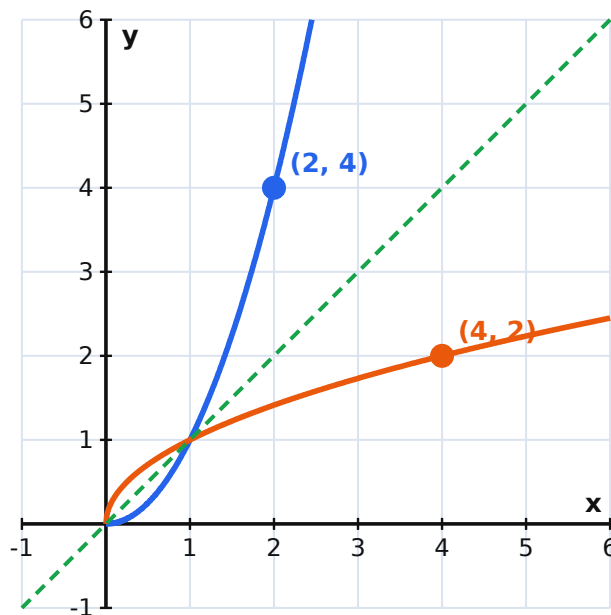
Swapping x and y reflects the graph over the line $y = x$. Every point (a, b) on f becomes (b, a) on f^{-1} . The domain of f becomes the range of f^{-1} , and the range of f becomes the domain of f^{-1} .



The point $(1, 3)$ on f matches $(3, 1)$ on the inverse.

What if the function is not one-to-one?

Restrict the domain to a piece that passes the horizontal line test. The function $f(x) = x^2$ is not one-to-one, but $f(x) = x^2$ with $x \geq 0$ is. Its inverse is $f^{-1}(x) = \sqrt{x}$, with domain $x \geq 0$.



With x at least 0, the square and the square root are inverses.

For a square-root function such as $f(x) = \sqrt{x-2}$ the inverse is $f^{-1}(x) = x^2 + 2$, but only for $x \geq 0$, because outputs of f are never negative.

Questions about inverse functions

Is $f^{-1}(x)$ the same as $1/f(x)$?

No. The exponent -1 here means inverse function, not reciprocal. For $f(x) = 2x + 1$, $f^{-1}(x) = (x - 1)/2$, while $1/f(x) = 1/(2x + 1)$.

How do you check that two functions are inverses?

Compute both compositions. If $f(g(x)) = x$ and $g(f(x)) = x$, the functions are inverses of each other.

Why does $y = x^2$ need a restricted domain?

Because 2 and -2 both give 4 , so one output has two inputs. Restricting to $x \geq 0$ makes it one-to-one, so the inverse is the square root.

What is the horizontal line test?

If every horizontal line crosses the graph at most once, the function is one-to-one and has an inverse. If some horizontal line crosses twice, it has no inverse on that domain.

How do the domain and range of an inverse relate to f ?

They swap places. The domain of f^{-1} equals the range of f , and the range of f^{-1} equals the domain of f .