

Precalculus - Polar Form and De Moivre's Theorem - Lesson

Common Core N-CN.B.4, N-CN.B.5, N-CN.B.6 · maths-pdf.fr/en/

The polar form of a complex number is $z = r(\cos \theta + i \sin \theta)$, where $r = |z|$ is the modulus and θ is the argument. Multiplying multiplies the moduli and adds the angles. De Moivre's theorem gives powers, $z^n = r^n (\cos n \theta + i \sin n \theta)$, and n evenly spaced n th roots.

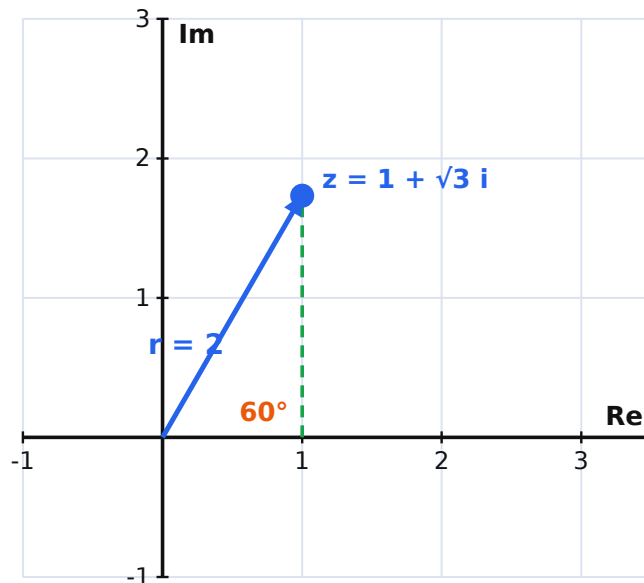
Precalculus · Common Core State Standards: **N-CN.B.4, N-CN.B.5, N-CN.B.6**

Key facts

- $z = a + bi = r(\cos \theta + i \sin \theta)$ with $r = \sqrt{a^2 + b^2}$ and $\tan \theta = b/a$.
- Product: multiply moduli, add arguments. Quotient: divide moduli, subtract arguments.
- De Moivre: $[r \operatorname{cis} \theta]^n = r^n \operatorname{cis}(n \theta)$.
- n th roots: $(r)^{1/n} \operatorname{cis}((\theta + 360 k)/n)$, $k = 0, \dots, n - 1$.
- The n roots form a regular n -gon on a circle of radius $r^{1/n}$.

What is the polar form of a complex number?

A complex number $z = a + bi$ is the point (a, b) in the complex plane. Its distance from the origin is the **modulus** $r = |z| = \sqrt{a^2 + b^2}$, and the angle from the positive real axis is the **argument** θ , with $\tan \theta = \frac{b}{a}$ in the correct quadrant. Because $a = r \cos \theta$ and $b = r \sin \theta$, the polar form is $z = r(\cos \theta + i \sin \theta)$, often written $r \operatorname{cis} \theta$.



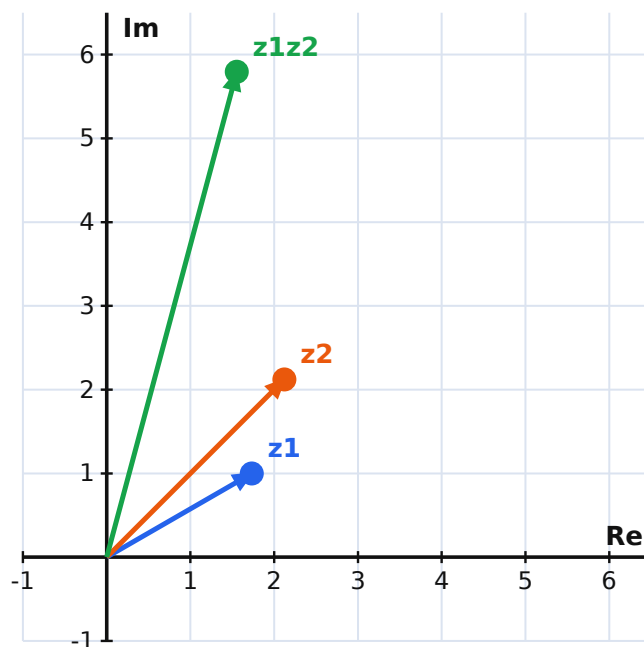
$z = 1 + \sqrt{3}i$ has modulus 2 and argument 60° , so $z = 2(\cos 60^\circ + i \sin 60^\circ)$.

How do you multiply and divide in polar form?

Product and quotient rules

If $z_1 = r_1 \operatorname{cis} \theta_1$ and $z_2 = r_2 \operatorname{cis} \theta_2$, then $z_1 z_2 = r_1 r_2 \operatorname{cis}(\theta_1 + \theta_2)$ and $\frac{z_1}{z_2} = \frac{r_1}{r_2} \operatorname{cis}(\theta_1 - \theta_2)$.

Multiplication multiplies the moduli and adds the angles. Geometrically it scales and rotates. For example, $2 \operatorname{cis} 30^\circ \cdot 3 \operatorname{cis} 45^\circ = 6 \operatorname{cis} 75^\circ$.



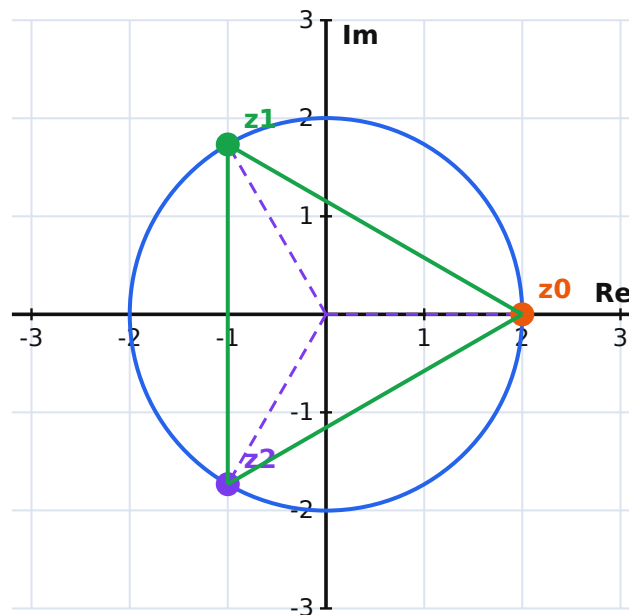
The product has modulus 2 times 3 = 6 and argument $30^\circ + 45^\circ = 75^\circ$.

What is De Moivre's theorem?

Repeated multiplication gives $[r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta)$. To compute $(1+i)^8$: $1+i = \sqrt{2} \operatorname{cis} 45^\circ$, so $(1+i)^8 = (\sqrt{2})^8 \operatorname{cis} 360^\circ = 16$.

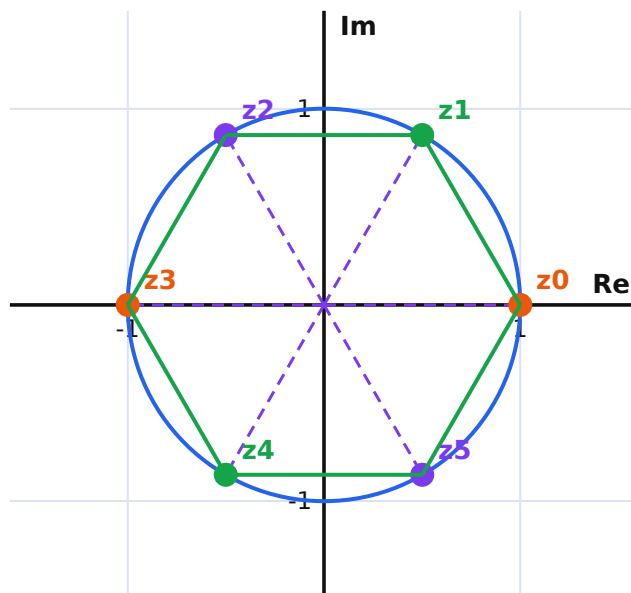
How do you find the n th roots of a complex number?

A nonzero number $z = r \operatorname{cis} \theta$ has exactly n distinct n th roots: $\sqrt[n]{r} \operatorname{cis} \left(\frac{\theta + 360^\circ k}{n} \right)$ for $k = 0, 1, \dots, n-1$. They lie on a circle of radius $\sqrt[n]{r}$, equally spaced $\frac{360^\circ}{n}$ apart. For the cube roots of 8, use $r = 8$, $\theta = 0^\circ$: $2 \operatorname{cis} 0^\circ$, $2 \operatorname{cis} 120^\circ$, $2 \operatorname{cis} 240^\circ$.



The three cube roots of 8 form an equilateral triangle on a circle of radius 2.

The roots of unity deserve special attention. The sixth roots of 1 are $\operatorname{cis} 0^\circ$, $\operatorname{cis} 60^\circ$, and so on in steps of 60° . They sit at the corners of a regular hexagon inscribed in the unit circle, and they add up to zero. Roots of unity appear in signal processing, in geometry (constructing regular polygons) and in solving equations of the form $z^n = 1$.



The sixth roots of unity form a regular hexagon.

Example (converting): To write $-3+3i$ in polar form, find $r = \sqrt{9+9} = 3\sqrt{2}$. The point lies in quadrant II, and $\arctan \frac{3}{-3} = -45^\circ$ would put it in quadrant IV, so add 180° to get $\theta = 135^\circ$. Always check the quadrant, because a calculator returns an angle between -90° and 90° only.

The steps are: write z in polar form, take the n th root of r , then divide the argument plus each multiple of 360° by n . The roots of 1, called roots of unity, form a regular polygon inscribed in the unit circle.

Questions about polar form and de moivre's theorem

How do you convert a complex number to polar form?

Find the modulus $r = \sqrt{a^2 + b^2}$ and the argument θ with $\tan \theta = b/a$, choosing the quadrant that matches the point. For $-3 + 3i$, $r = 3\sqrt{2}$ and $\theta = 135$ degrees.

Why do angles add when you multiply?

The product rule for cosine and sine uses the angle addition formulas, so multiplying $\text{cis } A$ by $\text{cis } B$ gives $\text{cis}(A + B)$. Geometrically, multiplying rotates by the other number's argument.

What is De Moivre's theorem used for?

It raises a complex number to a power quickly: take the power of the modulus and multiply the angle by the exponent. It also leads to the formula for n th roots.

How many n th roots does a complex number have?

Exactly n , all with the same modulus. Their arguments differ by 360 degrees divided by n .

What is $\text{cis } \theta$?

It is shorthand for $\cos \theta + i \sin \theta$, which saves writing when working with polar form.