

Precalculus - Rational Functions and Asymptotes

- Lesson

Common Core A-APR.D.7, F-IF.C.7d, A-CED.A.1 · maths-pdf.fr/en/

A rational function is a quotient of polynomials $p(x)/q(x)$. Zeros of the denominator that do not cancel give vertical asymptotes, and factors that cancel leave holes. Comparing degrees gives the end behavior: a horizontal asymptote when the numerator's degree is not larger, and a slant asymptote when it is exactly one larger.

Precalculus · Common Core State Standards: **A-APR.D.7, F-IF.C.7d, A-CED.A.1**

Key facts

- Domain: all real numbers except zeros of the denominator.
- Vertical asymptote: a denominator zero that does not cancel. Hole: a cancelled factor.
- Degree of numerator < denominator: $y = 0$. Equal degrees: $y =$ ratio of leading coefficients.
- Degree of numerator = degree of denominator + 1: slant asymptote from division.
- Always check rational equation answers for extraneous solutions.

What is a rational function?

A rational function is a quotient of two polynomials, $f(x) = \frac{p(x)}{q(x)}$ with $q(x) \neq 0$. Its domain excludes every zero of q . Rational functions can have breaks: vertical asymptotes, holes, and horizontal or slant asymptotes that describe the end behavior.

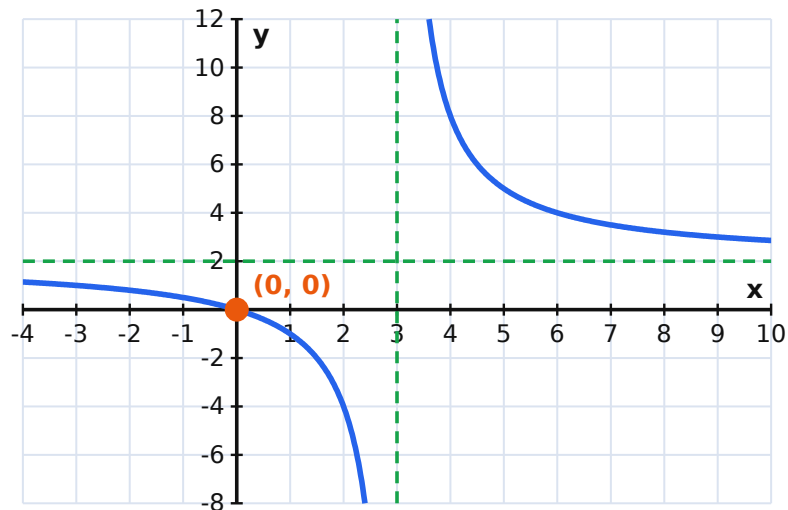
How do you find asymptotes and holes?

1. Factor numerator and denominator completely.
2. A factor that cancels gives a **hole** at that x -value.
3. A zero of the denominator that does not cancel gives a **vertical asymptote** there.
4. Compare the degrees of the numerator (n) and denominator (m) to find the end behavior.

Degrees	End behavior asymptote
$n < m$	Horizontal asymptote $y = 0$
$n = m$	Horizontal asymptote $y =$ ratio of leading coefficients

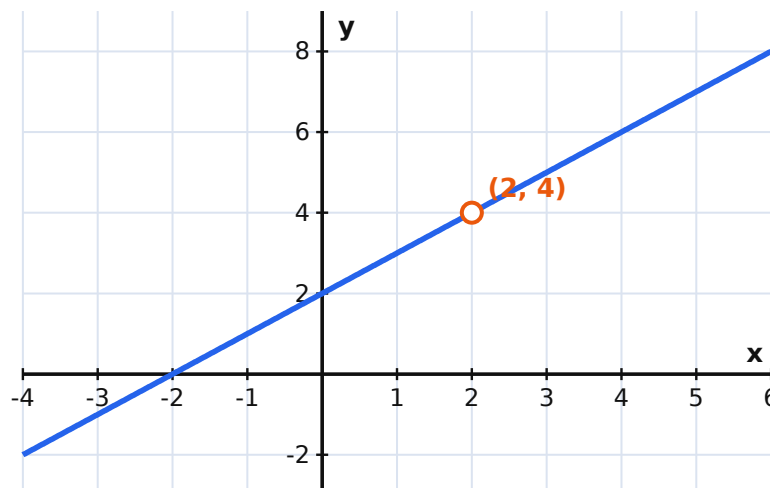
Degrees	End behavior asymptote
$n = m + 1$	Slant asymptote: the quotient (without the remainder) from division
$n > m + 1$	No horizontal or slant asymptote

Example: For $f(x) = \frac{2x}{x-3}$, the denominator is 0 at $x = 3$, so there is a vertical asymptote there. The degrees are equal and the leading coefficients are 2 and 1, so the horizontal asymptote is $y = 2$.



Dashed lines show the asymptotes $x = 3$ and $y = 2$.

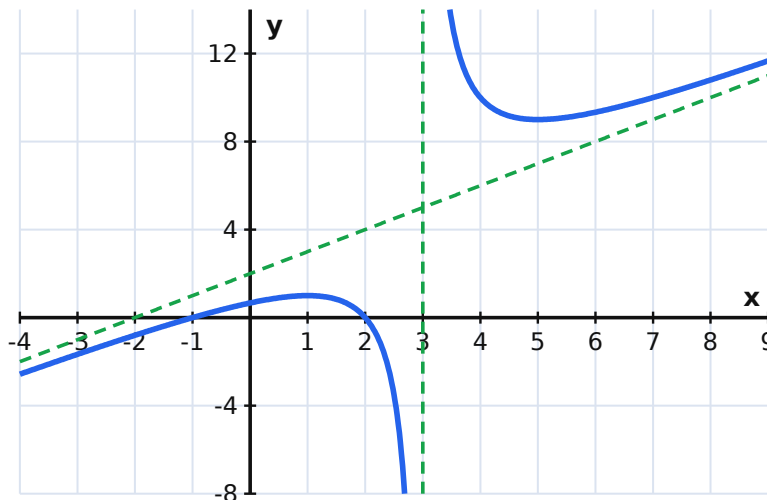
For $f(x) = \frac{x^2 - 4}{x - 2} = \frac{(x - 2)(x + 2)}{x - 2}$, the factor $x - 2$ cancels. The graph is the line $y = x + 2$ with a hole at $x = 2$, where $y = 4$.



A hole is a single missing point, drawn as an open circle.

How do you find a slant asymptote?

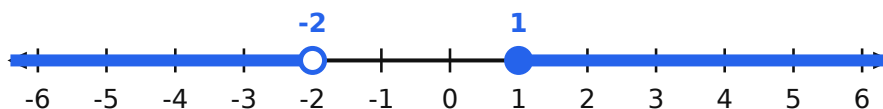
When the numerator's degree is one more than the denominator's, divide. For $\frac{x^2-x-2}{x-3}$, the quotient is $x+2$ with remainder 4, so $f(x) = x+2 + \frac{4}{x-3}$. As $|x|$ grows, the fraction goes to 0, and the graph approaches the line $y = x+2$.



A rational function can approach a slanted line.

How do you solve rational equations?

Multiply both sides by the least common denominator to clear fractions, solve, then check every answer in the original equation. A solution that makes a denominator 0 is **extraneous** and must be rejected. For rational inequalities, move everything to one side, factor, and use a sign chart with the zeros of the numerator and denominator.



Solution of $\frac{x-1}{x+2} \geq 0$: the zero 1 is included (closed) and the denominator zero -2 is excluded (open).

Intercepts: the x -intercepts are zeros of the numerator (that do not cancel), and the y -intercept is $f(0)$.

Questions about rational functions and asymptotes

What is the difference between a hole and a vertical asymptote?

Both come from a zero of the denominator. If the factor cancels with the numerator, the graph

has a hole (a single missing point). If it does not cancel, the graph has a vertical asymptote.

How do I find a horizontal asymptote?

Compare degrees. If the numerator has the lower degree, the asymptote is $y = 0$. If the degrees are equal, divide the leading coefficients. If the numerator has the higher degree, there is no horizontal asymptote.

Can a graph cross a horizontal asymptote?

Yes, in the middle of the graph. A horizontal asymptote describes the behavior as x gets very large or very small, not the behavior at every x .

What is an extraneous solution?

It is a value that appears when you solve but makes a denominator zero in the original equation. Always substitute your answers back and reject any that fail.

How do I find intercepts of a rational function?

Set the numerator equal to 0 for x -intercepts, ignoring values that also make the denominator 0. For the y -intercept, compute $f(0)$.